The Inadequacy of a Proposed Paraconsistent Set Theory

Abstract: We show that a paraconsistent set theory proposed in Weber (2010) is strong enough to provide a quite classical non-primitive notion of identity, so that the relation is an equivalence relation and also obeys full substitutivity: $a=b\rightarrow(F(a)\rightarrow F(b))$. With this as background it is shown that the proposed theory also proves $\forall x(x\neq x)$. While not by itself showing that the proposed system is trivial in the sense of proving all statements, it is argued that this outcome makes the system inadequate.

Weber (2010) proposes a paraconsistent set theory based upon a relevance logic with naïve abstraction. As the paper does not give a model for the proposed system, it must be assumed that the publication should be taken as posing a challenge to provide such a model, or else demonstrate that the system is trivial in the sense of proving all sentences. We do not show that the proposed system is trivial in the stated sense, but suggest that it is inadequate in the sense of proving theorems which one should not want as theorems even in a paraconsistent framework.

The system's primitive language has the connectives $\wedge$, $\sim$, $\rightarrow$, the identity sign, $\in$ for membership and the universal quantifier $\forall$. We will in our exposition use $\Bumpeq$ for Weber's form of identity, as we will show that a more appropriate notion of identity is supported in the suggested framework. Disjunction is introduced à la de Morgan, and the existential quantifier $\exists$ as expected. $A\leftrightarrow B$ is for $(A\rightarrow B)\wedge(B\rightarrow A)$. It is of importance that the arrow is not for material implication, but for some sort of relevant implication. The axiom schemas are as follows:

I	$A\rightarrow A$
IIa	$A\wedge B\rightarrow A$
IIb	$A\wedge B\rightarrow B$
III	$A\wedge(B\vee C)\rightarrow(A\wedge B)\vee(A\wedge C)$
IV	$(A\rightarrow B)\wedge(B\rightarrow C)\rightarrow(A\rightarrow C)$
V	$(A\rightarrow B)\wedge(A\rightarrow C)\rightarrow(A\rightarrow B\wedge C)$
VI	$(A\rightarrow\sim B)\rightarrow(B\rightarrow\sim A)$
VII	$\sim\sim A\rightarrow A$
VIII	$(A\rightarrow B)\rightarrow\sim(A\wedge\sim B)$

IXa $(A\rightarrow B)\rightarrow((B\rightarrow C)\rightarrow(A\rightarrow C))$

IXb $(A\rightarrow B)\rightarrow((C\rightarrow A)\rightarrow(C\rightarrow B))$

X $\forall xA\rightarrow A(y/x)$

XI $\forall x(A\rightarrow B)\rightarrow(A\rightarrow\forall xB)$

XII $\forall x(A\vee B)\rightarrow(A\vee\forall xB)$

The usual precautions are taken that in X, y is free for x in A, and that x is not free in A for axioms XI and XII.

Five rules and a so-called meta-rule are presupposed:

R1 $A, B\vdash A\wedge B$

R2 $A,A\rightarrow B\vdash B$

R3 $A\rightarrow B,C\rightarrow D\vdash(B\rightarrow C)\rightarrow(A\rightarrow D)$

R4 $A\vdash\forall xA$

R5 $x\Bumpeq y\vdash A(x)\rightarrow A(y)$

MR If $A\vdash B$ then $A\vee C\vdash B\vee C$

In addition to this, one has a naïve comprehension schema and extensionality:

Abstraction $\forall x(x\in\{y:A(y)\}\leftrightarrow A(x))$

Extensionality $\forall z(z\in x\leftrightarrow z\in y)\leftrightarrow x\Bumpeq y$

The author shows that $\Bumpeq$ is an equivalence relation, relying upon features of the arrow.

We first show that the proposed framework supports a more adequate and classical notion of identity. Define:

$$a=b=_D\forall z(a\in z\leftrightarrow b\in z)$$

The identity relation so taken is obviously an equivalence relation, and is suggested with a lore back to Leibniz. The proposed system proves the full substitution schema:

FS $\quad a=b\rightarrow(A(a)\rightarrow A(b))$

To see this, notice that $a=b\rightarrow(a\in\{x:A(x)\}\rightarrow b\in\{x:A(x)\})$ follows from an instance of axiom X, while also invoking IIa or IIb and axiom IV. From naïve abstraction we have that $A(a)\rightarrow a\in\{x:A(x)\}$ and $b\in\{x:A(x)\}\rightarrow A(b)$, so by invoking rule R3 we have that $(a\in\{x:A(x)\}\rightarrow b\in\{x:A(x)\})\rightarrow(A(a)\rightarrow A(b))$. FS follows by invoking axiom IV.

The identity sign needs not, as we have shown, be taken as primitive in a framework with naïve comprehension and such resources as in the system proposed.

In FS, a and b may fail to occur in the formula A(a), so FS supports the schema $a=a\rightarrow(p\rightarrow p)$ where p is any sentence. We investigate from this with the sentence $r\in r$, where r is Russell's paradoxical set $\{x:x\notin x\}$. In the proposed set up, $r\in r\wedge r\notin r$ is a theorem. By contraposing an instance of axiom VII we obtain the result that $\sim(r\in r\rightarrow r\in r)$ by using modus ponens, i.e. rule R2. By contraposing an instance of FS we have that $\sim(r\in r\rightarrow r\in r)\rightarrow a\neq a$. So $a\neq a$ by modus ponens. By generalization, rule R4, $\forall x(x\neq x)$ is a theorem of the proposed system.

If one understands the extensionality principle as stating that $x\Bumpeq y\rightarrow x=y$, it is easy to show, in light of our result, that all power-sets are paradoxical in the sense that they have members which also fail to be members: Since $x\neq x$ is a theorem, it follows that $\sim\forall y(y\in x\rightarrow y\in x)$, so $x\notin\{u:u\subset x]$. But also, $x\in\{u:u\subset x]$. It here seems worthwhile to bring up in passing that Weber (2010) does not show that any of his invoked infinite sets or power sets are not paradoxical in the sense just defined. But we would need such assurances in order to think that the proposed system can serve as a foundational theory in the way suggested.

It would not be reasonable to defend the proposed system by arguing that the identity relation we have isolated is not adequate. The notion of identity we have introduced wears its adequacy upon its sleeves, as it were. Instead, a proponent of the proposed system would be better served by somehow accepting as a mysterious outcome that all sets are self-identical while also self-*different*. The latter idea could in the framework e.g. be championed for on the ground that for any set a, $a \in \{x : r \in r\}$ while also $a \notin \{x : r \in r\}$, where, as above, r is Russell's paradoxical set. It is difficult to see, however, that such an attitude would not run counter to the stated goal of providing a foundation for standard mathematics. It is also difficult to see that a thesis to the effect that all objects, or sets, are different from themselves can be squared with our intuitions, and it would seem extraordinarily difficult to sell. It is strongly suggested that a theory that can appropriately deal with paradoxes, while also avoiding such controversial consequences for identity statements as pointed out, would be very much to be preferred.

Our note falls short of showing that the proposed set theory is trivial in the sense of proving all sentences. It shows, however, that the proposed system is strong enough to give an account of a quite classical notion of identity, and that it is inadequate because it has highly undesirable theorems concerning identity so understood.

It is not our burden to show that the proposed set theory is trivial, but rather its proponents' burden to show that it is not. Such a non-triviality proof for the system has not been forthcoming, and a model has not been provided.

References

Zach Weber(2010): Transfinite Numbers in Paraconsistent Set Theory. Review of Symbolic Logic 3 (1):71-92.